NIJENHUIS OPERATOR IN CONTACT HOMOLOGY AND DESCENDANT RECURSION IN SYMPLECTIC FIELD THEORY

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ABSTRACT. In this paper we investigate the algebraic structure related to a new type of correlator associated to the moduli spaces of S^1-parametrized curves in contact homology and rational symplectic field theory. Such correlators are the natural generalization of the non-equivariant linearized contact homology differential (after Bourgeois-Oancea) and give rise to an invariant Nijenhuis (or hereditary) operator (à la Magri-Fuchsteiner) in contact homology which recovers the descendant theory from the primaries. We also show how such structure generalizes to the full SFT Poisson homology algebra to a (graded symmetric) bivector. The descendant hamiltonians satisfy to recursion relations, analogous to bihamiltonian recursion, with respect to the pair formed by the natural Poisson structure in SFT and such bivector.

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INTRODUCTION

Starting from the early nineties, Boris Dubrovin and many of his collaborators have studied the relation between Gromov-Witten theory and the theory of integrable systems of PDEs, which was first noticed by Witten in [W]. The axioms of Frobenius manifold, cf. [D], encode all the properties that are satisfied by the algebraic structure generated by rational Gromov-Witten theory and that, in particular, generate an integrable Hamiltonian system of evolutionary PDEs. The structure of Frobenius manifold has proven to be central in many different areas of mathematics, from algebra to singularity theory, and provides for instance the most immediate approach to mirror symmetry.

One of the consequences of the axioms of a (homogenous) Frobenius manifold is that the associated Hamiltonian system of PDEs is actually bihamiltonian. Bihamiltonian structures where introduced by Magri in [M1] in the analysis of the so-called Lenard scheme (see e.g [GGKM]) to construct the KdV integrals. They consist of a manifold endowed with two Poisson tensors Π_1 and Π_2, mutually compatible in the sense that their Schouten-Nijenhuis bracket $[\Pi_1, \Pi_2]$ vanishes. Under the condition that the Poisson pencil $\Pi_\lambda = \Pi_2 - \lambda \Pi_1$ (a one-parametric family of Poisson tensors) has constant co-rank there exists a simple ricursive procedure for constructing a sequence of commuting integrals for both Poisson structures (see also [DZ] and the author's survey [R2]).

Theorem 0.1 ([M1]). *Let P be a manifold endowed with compatible Poisson tensors Π_1, Π_2 and associated Poisson brackets $\{\,\cdot\,,\,\cdot\,\}_1$, $\{\,\cdot\,,\,\cdot\,\}_2$. Let $k = \operatorname{corank} \Pi_1 = \operatorname{corank}(\Pi_1 + \epsilon\Pi_2)$ for arbitrary sufficiently small ϵ. Then the coefficients of the Taylor expansion*

$$c^\alpha(x, \lambda) = c^\alpha_{-1}(x) + \frac{c^\alpha_0(x)}{\lambda} + \frac{c^\alpha_1(x)}{\lambda^2} + \dots$$

of the Casimirs $c^\alpha(x, \lambda)$, $\alpha = 1, \dots, k$ of Poisson tensor Π_λ commute with respect to both Poisson brackets,

$$\{c^\alpha_i, c^\beta_j\}_{1,2} = 0 \ , \ i, j = -1, 0, 1, \dots.$$

Moreover $\{\,\cdot\,, c^\alpha_{i+1}\}_1 = \{\,\cdot\,, c^\alpha_i\}_2$, $i = -1, 0, 1, \dots.$

While in the literature this procedure is often called bihamiltonian recursion, we will use the term *bihamiltonian reconstruction*, to avoid confusion with the property of a sequence of symmetries $\{I_{\alpha,i}\}_{\alpha=1,\dots,k;i=-1,0,1,\dots}$ of being in *bihamiltonian recursion* if

$$\{\,\cdot\,, I_{\alpha,i}\}_2 = \sum_{\substack{\beta=1,\dots,k \\ j=0,\dots,i+1}} R^{\beta,j}_{\alpha,i}\{\,\cdot\,, I_{\beta,j}\}_1.$$

for some constant coefficients $R^{\beta,j}_{\alpha,i}$, $\alpha, \beta = 1, \dots, k$, $i, j = -1, 0, 1, \dots.$

In the context of Gromov-Witten theory and the associated Frobenius manifolds such reconstruction can be used to recover at least part of the symmetries for the Hamiltonian system. Actually, with great generality, given a homogeneous Frobenius manifold, finding a fundamental solution for the so called deformed flat connection (a special case of topological recursion relations for rational one-descendant GW invariants, see [DZ]) is a more powerful way to reconstruct the algebra of symmetries and, moreover, with this method the solution is automatically normalized to match the generating series of rational one-descendant GW invariants (often called J-function in the Gromov-Witten literature). Besides, the symmetries found this way are in bihamiltonian recursion anyway.

In contrast, bihamiltonian reconstruction is not, in general, directly related with enumerative geometry, which results in discrepancies with the J-function. Even worse, even when the hypothesis of the above theorem are satisfied, it might happen that the two Poisson tensors have some Casimir in common. This means that, when starting with such common Casimirs as $c^\alpha_{-1}(x)$, in the above Taylor expansion, all of the other coefficients trivially vanish. This is precisely what happens in the case of the Gromov-Witten theory of the projective line $\mathbb{P}^1$, where bihamiltonian recursion is only capable of recovering the symmetries generated by descendants of the Kähler class, but not those of the unity class. This pathology of the Poisson

pencil is called *resonance*.

This paper deals with symplectic field theory (SFT) and a recursion procedure for descendants that has much in common with bihamiltonian recursion in Gromov-Witten theory. Introduced by H. Hofer, A. Givental and Y. Eliashberg in 2000 [EGH], SFT is a very large project and can be viewed as a topological quantum field theory approach to Gromov-Witten theory. Besides providing a unified view on established pseudoholomorphic curve theories like symplectic Floer homology, contact homology and Gromov-Witten theory, it sheds considerable light on the appearence of infinite dimensional Hamiltonian systems in the theory of holomorphic curves (see [R2] for an review on this topic which includes SFT).

Indeed, symplectic field theory leads to algebraic invariants with very rich algebraic structures and in particular, as it was pointed out by Eliashberg in his ICM 2006 plenary talk ([E]), the integrable systems of rational Gromov-Witten theory very naturally appear in rational symplectic field theory by using the link between the rational symplectic field theory of prequantization spaces in the Morse-Bott version and the rational Gromov-Witten potential of the underlying symplectic manifold (see the recent papers [R1], [R2]). After introducing gravitational descendants (see [F2]) along the lines of Gromov-Witten theory, it is precisely the natural algebraic structure of SFT that provides a natural link between holomorphic curves and (quantum) integrable systems.

In this paper we explore the potentiality of an intrinsic difference between Gromov-Witten and symplectic field theory: the moduli spaces of holomorphic maps studied by the latter carry special evaluation maps controlling the relative gluing angle of different components of a multi-floor configuration (the SFT generalization of nodal curves, see [EGH]). These can be used to define new correlators that were not present in the original theory, but give interesting recursive formulas for one-point descendants (and probably beyond), which are similar but not equivalent to bihamiltonian reconstruction, topological recursion and, ultimately, to the integrability properties of the SFT infinite dimensional hamiltonian system.

In many senses these extra correlators which control the gluing angles of different components of curves in the boundary strata are a natural generalization of the non-S^1-equivariant differential for the non-equivariant linearized contact homology of Bourgeois-Oancea, [BO] (see also [FR1]), to non-linearized contact homology and full rational SFT. With this in mind we conclude that the non-equivariant differential plays a role similar to the second Poisson structure of Gromov-Witten theory with respect to gravitational descendants. More precisely we show how our generalized non-equivariant differential (the potential encoding such new correlators), denoted by by N, satisfies (in contact homology) the axioms of a Nijenhuis operator.

Recall from Magri and Fuchsteiner, [M2][Fu], that on a manifold M, a Nijenhuis (or hereditary) operator $N \in \mathcal{T}^{(1,1)}M$ (where $\mathcal{T}^{(k,l)}M$ denotes the space of (k,l)-tensor fields on M) is one whose Nijenhuis torsion vanishes:

$$N_b^a \left(\frac{\partial N_c^b}{\partial x^d} - \frac{\partial N_d^b}{\partial x^c} \right) - \frac{\partial N_c^a}{\partial x^b} N_d^b + \frac{\partial N_d^a}{\partial v^b} N_c^b = 0$$

This condition ensures that, given a sequence of commuting vector fields

$$X_{\alpha,0} \in \mathcal{T}^{(1,0)}M, \qquad [X_{\alpha,0}, X_{\beta,0}] = 0, \qquad \alpha, \beta = 1, \dots, n$$

that are also symmetries of the operator N, i.e. $\mathcal{L}_{X_{\alpha,0}} N = 0$, we can enlarge the commuting system by recursively applying the operator N:

$$X_{\alpha,k} := N^k(X_{\alpha,0}), \qquad [X_{\alpha,k}, X_{\beta,j}] = 0, \qquad k,j \in \mathbb{N}.$$

Given a Poisson tensor $\Pi \in \mathcal{T}^{(2,0)} M$ on M, in [MM] Magri and Morosi further studied the compatibility conditions of a Nijenhuis operator N with the Poisson structure. N is said to be compatible with Π if the following two conditions hold:

$$N \circ \Pi = \Pi \circ {}^t N$$

$$\Pi^{lj} \left(\frac{\partial N_m^k}{\partial x^l} - \frac{\partial N_l^k}{\partial x^m} \right) - \Pi^{kl} \frac{\partial N_m^j}{\partial x^l} - N_m^l \frac{\partial \Pi^{kj}}{\partial x^l} + N_l^j \frac{\partial \Pi^{kl}}{\partial x^m} = 0$$

When these equations are satisfied, the pair (Π, N) is called a Poisson-Nijenhuis structure on M. The main property, then, is that one can define a sequence of $(2,0)$-tensors $\Pi_k = N^k \circ \Pi$, $k = 0, 1, 2, \ldots$ which are Poisson and are pairwise compatible in the sense that they pairwise form Poisson pencils. In particular we get the bihamiltonian structure $(\Pi_0 = \Pi, \Pi_1 = N \circ \Pi)$.

The theory of Poisson-Nijenhuis structure is very well developed and its relation with integrability is deep. In particular the study of the spectrum of N plays a fundamental role, in that the eigenvalues of N form a system of commuting symmetries in bihamiltonian recursion (see for instance [MM],[KSM],[DLF]).

The first part of the paper deals with the contact homology case. In contact homology, thanks to the total absence of non-constant nodal configurations (due to the maximum principle for holomorphic curves in symplectizations), we prove that the knowledge of N and of the primary theory (contact homology differential X, with no descendants) is sufficient to completely reconstruct the descendant vector fields $X_{\alpha,n}$ as differential operators in the variables associated to Reeb orbits on contact homology.

In the case of full rational SFT (of any target stable Hamiltonian structure), our bihamiltonian recursion is slightly less effective because of the presence of non-constant nodal curves. Formally the result is similar, i.e. the descendant Hamiltonians $\mathbf{h}_{\alpha,n}$ satisfy recursion relations which are completely analogus to bihamiltonian recursion for a pair of bivectors Π, ω, where Π is the natural Poisson structure on the SFT homology algebra and ω is a *graded symmetric* even bivector which is the SFT-generalization of N.

In any case this looks like a fundamental step in understanding the relation between completeness of the contact homology vector field system, or the SFT Hamiltonian system, and the underlying symplectic topology of the target manifold. Indeed the full information is contained in the non-equivariant correlators forming N and ω, and we plan to study the consequences in a subsequent publication.

I would like to thank the American Institute of Mathematics for allowing me to organize the workshop on *Integrable systems in Gromov-Witten and symplectic field theory*, in January 2012, during which I could speak to Boris Dubrovin, who then noticed how my equations for N where equivalent to vanishing of Nijenhuis torsion and who pointed me to the correct reference (Magri, Morosi and Fuchsteiner) to see the big picture.

1. Notions from Symplectic Field Theory

Symplectic field theory (SFT), introduced by Y. Eliashberg, A. Givental and H. Hofer in [EGH], consists in a unified and comprehensive approach to the theory of holomorphic curves in symplectic and contact topology. In the spirit of a topological field theory, it assigns algebraic invariants to closed manifolds with a stable Hamiltonian structure. We recall here the main ideas from [EGH],[FR],[FR1].

1.1. Stable Hamiltonian structures.
A Hamiltonian structure (see [BEHWZ]) on a closed $(2n-1)$-dimensional manifold V is a closed two-form Ω on V of maximal rank $2n - 2$. This means that $\ker \Omega = \{v \in TV : \Omega(v, \cdot) = 0\}$ is a one-dimensional distribution. A Hamiltonian structure is called stable if there exists a one-form λ and a vector field R (called Reeb vector field) on V such that R generates $\ker \Omega$, $\lambda(R) = 1$ and $\iota_R d\lambda = 0$. Notice that, this way, $\xi = \ker \lambda$ is a symplectic hyperplane distribution. Also, notice that if R exists, it is completely determined by Ω and λ.

Example 1.1. Any contact form λ on V provides a stable Hamiltonian structure $(\Omega := d\lambda, \lambda)$ on V where the symplectic hyperplane distribution coincides with the contact structure.

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Example 1.2. Given a principal circle bundle $\pi : V \to M$ over a closed symplectic manifold (M, Ω_M) and any connection 1-form λ, $(\Omega = \pi^*\Omega_M, \lambda$ is a stable Hamiltonian structure on V.

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Example 1.3. Given a closed symplectic manifold (M, Ω_M) and a symplectomorphism $\phi \in \mathrm{Symp}(M, \Omega_M)$, consider the symplectic mapping torus $V = M_\phi = \mathbb{R} \times M / \{(t, p) \sim (t+1, \phi(p))\}$. The natural splitting $TV = TS^1 \oplus TM$ allows to define the lift Ω of Ω_M to V. Then $(\Omega, \lambda = dt)$, where t is the natural S^1-coordinate, is a stable Hamiltonian structure (with integrable symplectic distribution $\ker \lambda$).

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Given a stable Hamiltonian structure (V, Ω, λ), consider a complex structure $\underline{J}_\xi$ on the hyperplane ditribution $\xi = \ker \lambda$ which is $\Omega|_\xi$-compatible (i.e. $\Omega|_\xi(\cdot, \underline{J}_\xi \cdot)$ is a metric on ξ). Such complex structures form a non-empty, contractible set. We extend $\underline{J}_\xi$ uniquely from ξ to an almost complex structure $\underline{J}$ on the cylinder $\mathbb{R} \times V$ by requiring that $\underline{J}$ is $\mathbb{R}$-invariant and $\underline{J}\partial_s = R$, ∂_s being the $\mathbb{R}$-direction.

1.2. Symplectic field theory.
Symplectic field theory assigns algebraic invariants to closed manifolds V with a stable Hamiltonian structure. The invariants are defined by counting $\underline{J}$-holomorphic curves in $\mathbb{R} \times V$ with finite energy.

Let us recall the definition of moduli spaces of holomorphic curves studied in rational SFT. Let Γ^+, Γ^- be two ordered sets of closed orbits γ of the Reeb vector

field R on V, i.e., $\gamma : \mathbb{R} \to V$, $\gamma(t + T) = \gamma(t)$, $\dot{\gamma} = R$, where $T > 0$ denotes the period of γ. Here we assume that the stable Hamiltonian structure is Morse in the sense that all closed orbits of the Reeb vector field are nondegenerate in the sense of [BEHWZ]; in particular, the set of closed Reeb orbits is discrete. Given a closed Reeb orbit γ of any multiplicity, we will denote by $\bar{\gamma}$ its underlying simple Reeb orbit. At each simple Reeb orbit we will fix a closed form $d\phi_{\bar{\gamma}}$ generating $H^1(S^1)$.

Then the (parametrized) moduli space $\mathcal{M}^0_{r,A}(\Gamma^+, \Gamma^-)$ consists of tuples $(u, (z_k^{\pm}), (z_i))$, where $(z_k^{\pm}), (z_i)$ are three disjoint ordered sets of points on $\mathbb{CP}^1$, which are called positive and negative punctures, and additional marked points, respectively. We will also fix an asymptotic marker at each puncture, i.e. a ray originating at the puncture. The map $u : \dot{S} \to \mathbb{R} \times V$ starting from the punctured Riemann surface $\dot{S} = \mathbb{CP}^1 - \{(z_k^{\pm})\}$ is required to satisfy the Cauchy-Riemann equation

$$\bar{\partial}_{\underline{J}} u = du + \underline{J}(u) \cdot du \cdot i = 0$$

with the complex structure i on $\mathbb{CP}^1$. Assuming we have chosen cylindrical coordinates $\psi_k^{\pm} : \mathbb{R}^{\pm} \times S^1 \to \dot{S}$ around each puncture $z_k^{\pm}$ in the sense that $\psi_k^{\pm}(\pm\infty, t) = z_k^{\pm}$, the map u is additionally required to show for all $k = 1, ..., n^{\pm}$ the asymptotic behaviour

$$\lim_{s \to \pm\infty} (u \circ \psi_k^{\pm})(s, t + t_0) = (\pm\infty, \gamma_k^{\pm}(T_k^{\pm} t))$$

with some $t_0 \in S^1$ and the orbits $\gamma_k^{\pm} \in \Gamma^{\pm}$, where $T_k^{\pm} > 0$ denotes period of $\gamma_k^{\pm}$. In order to assign an absolute homology class A to a holomorphic curve $u : \dot{S} \to \mathbb{R} \times V$ we have to employ spanning surfaces u_γ connecting a given closed Reeb orbit γ in V to a linear combination of circles c_s representing a basis of $H_1(V)$,

$$\partial u_\gamma = \gamma - \sum_s n_s \cdot c_s$$

in order to define

$$A = [u_{\Gamma^+}] + [u(\dot{S})] - [u_{\Gamma^-}],$$

where $[u_{\Gamma^{\pm}}] = \sum_{n=1}^{s^{\pm}} [u_{\gamma_n^{\pm}}]$ viewed as singular chains.

Observe that the group $\text{Aut}(\mathbb{CP}^1)$ of Moebius transformations acts on elements in $\mathcal{M}^0 = \mathcal{M}^0_{r,A}(\Gamma^+, \Gamma^-)$ in an obvious way,

$$\varphi.(u, (z_k^{\pm}), (z_i)) = (u \circ \varphi^{-1}, (\varphi(z_k^{\pm})), (\varphi(z_i))), \quad \varphi \in \text{Aut}(\mathbb{CP}^1),$$

and we obtain the moduli space $\mathcal{M} = \mathcal{M}_{r,A}(\Gamma^+, \Gamma^-)$ studied in symplectic field theory by dividing out this action and the natural $\mathbb{R}$-action on the target manifold $(\mathbb{R} \times V, \underline{J})$. Furthermore it was shown in [BEHWZ] that this moduli space can be compactified to a moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$ by adding moduli spaces of multi-floor curves with nodes. In particular, the moduli space has codimension-one boundary given by (fibre) products $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{r_1, A_1}(\Gamma_1^+, \Gamma_1^-) \times \overline{\mathcal{M}}_{r_2, A_2}(\Gamma_2^+, \Gamma_2^-)$ of lower-dimensional moduli spaces.

Let us now briefly introduce the algebraic formalism of rational SFT as described in [EGH]:

Let us fix a trivialization of the symplectic bundle $(\xi, \Omega|_{\xi})$ over each curve C_i. This induces a trivialization a hoomotopically unique trivialization of the same bundle over each periodic Reeb orbit γ via the spanning surface u_γ. Let us use this trivialization to define the Conley-Zehnder index of the Reeb orbit (the Maslov index of the path in $Sp(2m - 2, \mathbb{R})$ given by the linearized Reeb flow along γ). Recall that a multiply-covered Reeb orbit γ^k is called bad if $\text{CZ}(\gamma^k) \neq \text{CZ}(\gamma)$

mod 2, where $\mathrm{CZ}(\gamma)$ denotes the Conley-Zehnder index of γ. Calling a Reeb orbit γ *good* if it is not bad, denote by $\mathcal{P}$ the space of good Reeb orbits. We assign to every good Reeb orbit γ two formal graded variables p_γ, q_γ with grading

$$|p_\gamma| = m - 3 - \mathrm{CZ}(\gamma), |q_\gamma| = m - 3 + \mathrm{CZ}(\gamma)$$

when $\dim V = 2m - 1$.

Assuming we have chosen a basis $A_0, \ldots, A_M$ of $H_2(V)$, we assign to every A_i a formal variable z_i with grading $|z_i| = -2c_1(A_i)$. In order to include higher-dimensional moduli spaces we further assume that a string of closed (homogeneous) differential forms $\Theta = (\theta_1, \ldots, \theta_N)$ on V is chosen and assign to every $\theta_\alpha \in \Omega^*(V)$ a formal variables t^α with grading

$$|t^\alpha| = 2 - \deg \theta_\alpha.$$

With this let $\mathfrak{P}$ be the Poisson algebra of formal power series in the variables p_γ and t_i with coefficients which are polynomials in the variables q_γ and Laurent series in the Novikov ring variables z_n, with Poisson bracket given by

$$\{f, g\} = \sum_\gamma \kappa_\gamma \Big(\frac{\partial f}{\partial p_\gamma} \frac{\partial g}{\partial q_\gamma} - (-1)^{|f||g|} \frac{\partial g}{\partial p_\gamma} \frac{\partial f}{\partial q_\gamma} \Big).$$

where κ_γ is the multiplicity of the orbit γ.

Consider the union $\overline{\mathcal{M}}_{r,n^+,n^-,A}$ of all moduli spaces $\overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$ for $|\Gamma^+| = n^+$ and $|\Gamma^-| = n^-$. As in Gromov-Witten theory we want to organize all moduli spaces $\overline{\mathcal{M}}^A_{r,n^+,n^-}$ into a generating function $\mathbf{h} \in \mathfrak{P}$, called *Hamiltonian*. In order to include also higher-dimensional moduli spaces, in [EGH] the authors follow the approach in Gromov-Witten theory to integrate the chosen differential forms θ_α and $[\gamma]$ (the canonical basis of $H^*(\mathcal{P})$) over the moduli spaces after pulling them back under the evaluation maps ev_i, $i = 1, \ldots, r$, $ev\pm, j$, $j = 1, \ldots, n^\pm$ at the marked points and punctures to the target manifold V and the space of (positive or negative) good Reeb orbits $\mathcal{P}$, respectively. Consider furthermore evaluation maps $\mathrm{ev} \pm\infty, j : \overline{\mathcal{M}} \to \bigcup_{\gamma \in \mathcal{P}} \bar\gamma$, $j = 1, \ldots, n^\pm$ defined by the asymptotic markers at each puncture. Let

$$t = \sum_{\alpha=1}^N t^\alpha \theta_\alpha$$

$$p = \sum_{\gamma \in \mathcal{P}} \frac{1}{\kappa_\gamma} p_\gamma [\gamma]$$

$$q = \sum_{\gamma \in \mathcal{P}} \frac{1}{\kappa_\gamma} q_\gamma [\gamma]$$

The Hamiltonian $\mathbf{h}$ is then defined by

$$\mathbf{h} = \sum_{\substack{r, A, \\ n^+, n^-}} \frac{1}{r! n^+! n^-!} \int_{\overline{\mathcal{M}}_{r,n^+,n^-,A}/\mathbb{R}} \bigwedge_{i=1}^r \mathrm{ev}_i^* \, t \bigwedge_{j=1}^{n^+} (\mathrm{ev}_{+,j}^* \, p \wedge \mathrm{ev}_{+\infty,j}^* \, d\phi_{\bar\gamma_j^+}) \wedge$$

$$\bigwedge_{k=1}^{n^-} (\mathrm{ev}_{-,k}^* \, q \wedge \mathrm{ev}_{-\infty,k}^* \, d\phi_{\bar\gamma_k^-}) \, z^A$$

with $z^A = z_0^{d_0} \cdot \ldots \cdot z_M^{d_M}$ for $A = d_0 A_0 + \ldots + d_M A_M$.

1.3. **Gravitational descendants.** We recall the definition of gravitational descendants in symplectic field theory (see [F2]). In complete analogy to Gromov-Witten theory we can introduce r tautological line bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$ over each moduli space $\overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$, as the pull-back of the relative dualizing sheaf of $\pi_i : \overline{\mathcal{M}}_{r+1,A}(\Gamma^+, \Gamma^-) \to \overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$ under the canonical section $\sigma_i : \overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-) \to \overline{\mathcal{M}}_{r+1,A}(\Gamma^+, \Gamma^-)$ mapping to the i-th marked point in the fibre.

As in Gromov-Witten theory we would like to consider the integration of (powers of) the first Chern class of the tautological line bundles over the moduli space, which by Poincaré duality corresponds to counting common zeroes of sections of such bundles. However, in symplectic field theory the moduli spaces can have codimension-one boundary, so we need to replace integration of the first Chern class of the tautological line bundle over a single moduli space with a construction involving all moduli space at once, which preserves the algebraic structure of SFT.

Following the compactness statement in [BEHWZ], the codimension-one boundary of a moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$ of SFT holomorphic curves consists of curves with two levels (in the sense of [BEHWZ]). More precisely, each component of the boundary has the form of a fibred product $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{r_1,A_1}(\Gamma_1^+, \Gamma_1^-)$ $_{(\mathrm{ev}_{-,n_1^-}, \mathrm{ev}_{-\infty,n_1^-})} \times_{(\mathrm{ev}_{+,1}, \mathrm{ev}_{+\infty,1})} \overline{\mathcal{M}}_{r_2,A_2}(\Gamma_2^+, \Gamma_2^-)$ of moduli spaces (of strictly lower dimension), where the marked points distribute on the two levels. Consider a boundary component where the i-th marked point sits, say, on the first level $\overline{\mathcal{M}}_1$: it directly follows from the definition of the tautological line bundle $\mathcal{L}_i$ at the i-th marked point over $\overline{\mathcal{M}}$ that, over such boundary component,

$$\mathcal{L}_i \big|_{\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2} = \pi_1^* \mathcal{L}_{i,1}$$

where $\mathcal{L}_{i,1}$ denotes the tautological line bundle over the moduli space $\overline{\mathcal{M}}_1$ and $\pi_1 : \overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 \to \overline{\mathcal{M}}_1$ is the projection onto the first factor. With this we can now give the definition of coherent collections of sections in tautological line bundles from [F2].

Definition 1.4. *Assume that we have chosen sections s_i in the tautological line bundles $\mathcal{L}_i$ over all moduli spaces $\overline{\mathcal{M}}$ of $\underline{J}$-holomorphic curves of SFT. Then these collections of sections (s_i) are called* coherent *if for every section s_i of $\mathcal{L}_i$ over a moduli space $\overline{\mathcal{M}}$ the following holds: over each codimension-one boundary component $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2$ of $\overline{\mathcal{M}}$, the section s_i agrees with the pull-back $\pi_1^* s_{i,1}$ ($\pi_2^* s_{i,2}$) of the chosen section $s_{i,1}$ ($s_{i,2}$) of the tautological line bundle $\mathcal{L}_{i,1}$ over $\overline{\mathcal{M}}_1$ ($\mathcal{L}_{i,2}$ over $\overline{\mathcal{M}}_2$), assuming that the i-th marked point sits on the first (second) level.*

Since in the end we will again be interested in the zero sets of these sections, we will assume that all occuring sections are sufficiently generic, in particular, transversal to the zero section. Furthermore, we want to assume that all the chosen sections are indeed invariant under the obvious symmetries like reordering of punctures and marked points. In order to meet both requirements, it follows that actually need to employ multi-sections (in the sense of branched manifolds). On the other hand, it is clear that one can always find coherent collections of (transversal) sections (s) by using induction on the dimension of the underlying moduli space.

For every tuple $(j_1, \dots, j_r)$ of natural numbers we choose j_i coherent collections of sections $(s_{i,k})$ of $\mathcal{L}_i$. Then we define for every moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$,

$$\overline{\mathcal{M}}^{(j_1, \dots, j_r)} = s_{1,1}^{-1}(0) \cap \dots \cap s_{1,j_1}^{-1}(0) \cap \dots \cap s_{r,1}^{-1}(0) \cap \dots \cap s_{r,j_r}^{-1}(0) \subset \overline{\mathcal{M}}.$$

Note that by choosing all sections sufficiently generic, we can assume $\overline{\mathcal{M}}^{(j_1,\ldots,j_r)} = \overline{\mathcal{M}}^{(j_1,\ldots,j_r)}_{r,A}(\Gamma^+,\Gamma^-)$ is a branched-labelled submanifold of the moduli space $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$. Note that by definition

$$\overline{\mathcal{M}}^{(j_1,\ldots,j_r)} = \overline{\mathcal{M}}^{(j_1,0,\ldots,0)} \cap \ldots \cap \overline{\mathcal{M}}^{(0,\ldots,0,j_r)},$$

and it follows from the coherency condition that the codimension-one boundary of $\overline{\mathcal{M}}^{(0,\ldots,0,j,0,\ldots,0)}$ is given by the products $\overline{\mathcal{M}}^{(0,\ldots,0,j,0,\ldots,0)}_1 \times \overline{\mathcal{M}}_2$ or $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}^{(0,\ldots,0,j,0,\ldots,0)}_2$ (depending on the i-th marked points sitting on the first or second level).

With this we can define the descendant Hamiltonian of SFT, which we will denote by $\tilde{\mathbf{h}}$, while the Hamiltonian $\mathbf{h} = \tilde{\mathbf{h}}|_{t^{\alpha,j}=0,j>0}$ defined in [EGH] will from now on be called *primary*. In order to keep track of the descendants we will assign to every chosen differential form θ_α now a sequence of formal variables $t^{\alpha,j}$ with grading

$$|t^{\alpha,j}| = 2(1-j) - \deg\theta_\alpha$$

and form an extended Poisson algebra $\tilde{\mathfrak{P}}$ accordingly. Then the descendant Hamiltonian $\tilde{\mathbf{h}} \in \tilde{\mathfrak{P}}$ of (rational) SFT is defined by

$$\tilde{\mathbf{h}} = \sum_{\substack{r,A,I \\ n^+,n^-}} \int_{\overline{\mathcal{M}}^{(j_1,\ldots,j_r)}_{r,n^+,n^-,A}/\mathbb{R}} \mathrm{ev}_1^*\,\theta_{\alpha_1} \wedge \ldots \wedge \mathrm{ev}_r^*\,\theta_{\alpha_r} \bigwedge_{j=1}^{n^+}(\mathrm{ev}_{+,j}^*\,p \wedge \mathrm{ev}_{+\infty,j}^*\,d\phi_{\bar{\gamma}_j^+})\wedge$$

$$\bigwedge_{k=1}^{n^-}(\mathrm{ev}_{-,k}^*\,q \wedge \mathrm{ev}_{-\infty,k}^*\,d\phi_{\bar{\gamma}_k^-})\,t^I z^A,$$

where $t^I = t^{\alpha_1,j_1}\ldots t^{\alpha_r,j_r}$ and $z^A = z_0^{d_0}\cdot\ldots\cdot z_M^{d_M}$ for $A = d_0 A_0 + \ldots + d_M A_M$.

1.4. **Hamiltonian systems with symmetries.** Symplectic field theory assigns to every contact manifold not only a Poisson algebra, rational SFT homology, but also, thanks to gravitational descendants, a Hamiltonian system in it with an infinite number of symmetries.

Theorem 1.5. *Differentiating the rational Hamiltonian $\tilde{\mathbf{h}} \in \tilde{\mathfrak{P}}$ with respect to the formal variables $t_{\alpha,p}$ defines a sequence of classical Hamiltonians*

$$\tilde{\mathbf{h}}_{\alpha,p} = \frac{\partial\tilde{\mathbf{h}}}{\partial t^{\alpha,p}} \in H_*(\tilde{\mathfrak{P}},\{\tilde{\mathbf{h}},\cdot\})$$

in the rationall SFT homology algebra with differential $\tilde{d} = \{\tilde{\mathbf{h}},\cdot\} : \tilde{\mathfrak{P}} \to \tilde{\mathfrak{P}}$, which commute with respect to the bracket on $H_*(\tilde{\mathfrak{P}},\{\tilde{\mathbf{h}},\cdot\})$,

$$\{\tilde{\mathbf{h}}_{\alpha,p},\tilde{\mathbf{h}}_{\beta,q}\} = 0,\ (\alpha,p),(\beta,q) \in \{1,\ldots,N\}\times\mathbb{N}.$$

Note that everything is an immediate consequence of the master equation $\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}\} = 0$, which can be proven in the same way as in the case without descendants using the results in [F2]. While the boundary equation $\tilde{d} \circ \tilde{d} = 0$, $\tilde{d} = \{\tilde{\mathbf{h}},\cdot\}$ is well-known to follow directly from the identity $\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}\} = 0$, the fact that every $\tilde{\mathbf{h}}_{\alpha,p}$, $(\alpha,p) \in \{1,\ldots,N\}\times\mathbb{N}$ defines an element in the homology $H_*(\tilde{\mathfrak{P}},\{\tilde{\mathbf{h}},\cdot\})$ follows from the identity

$$\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}_{\alpha,p}\} = 0,$$

which can be shown by differentiating the master equation with respect to the $t^{\alpha,p}$-variable and using the graded Leibniz rule,

$$\frac{\partial}{\partial t^{\alpha,p}}\{f,g\} = \{\frac{\partial f}{\partial t^{\alpha,p}},g\} + (-1)^{|t^{\alpha,p}||f|}\{f,\frac{\partial g}{\partial t^{\alpha,p}}\}.$$

On the other hand, in order to see that any two $\tilde{\mathbf{h}}_{\alpha,p}$, $\tilde{\mathbf{h}}_{\beta,q}$ commute *after passing to homology* it suffices to see that by differentiating twice (and using that all summands in $\mathbf{h}$ have odd degree) we get the identity

$$\{\tilde{\mathbf{h}}_{\alpha,p},\tilde{\mathbf{h}}_{\beta,q}\} + (-1)^{|t^{\alpha,p}|}\{\tilde{\mathbf{h}},\frac{\partial^2\tilde{\mathbf{h}}}{\partial t^{\alpha,p}\partial t^{\beta,q}}\} = 0.$$

We now turn to the question of independence of these nice algebraic structures from the choices like contact form, cylindrical almost complex structure, representatives for the classes $[\theta_\alpha] \in H^*(V)$ and $[d\phi_{\bar\gamma}] \in H^*(S^1)$, abstract polyfold perturbations and, of course, the choice of the coherent collection of sections. This is the content of the following theorem proven in [F2].

Theorem 1.6. *For different choices of contact form $\lambda^\pm$, cylindrical almost complex structure $\underline{J}^\pm$, representatives for the classes $[\theta_\alpha] \in H^*(V)$ and $[d\phi_{\bar\gamma}] \in H^*(S^1)$, abstract polyfold perturbations and sequences of coherent collections of sections $(s_j^\pm)$ the resulting systems of commuting functions $\tilde{\mathbf{h}}_{\alpha,p}^-$ on $H_*(\tilde{\mathfrak{P}}^-,d^-)$ and $\tilde{\mathbf{h}}_{\alpha,p}^+$ on $H_*(\tilde{\mathfrak{P}}^+,\tilde{d}^+)$ are isomorphic, i.e. there exists an isomorphism of the Poisson algebras $H_*(\tilde{\mathfrak{P}}^-,\tilde{d}^-)$ and $H_*(\tilde{\mathfrak{P}}^+,\tilde{d}^+)$ which maps $\tilde{\mathbf{h}}_{\alpha,p}^- \in H_*(\tilde{\mathfrak{P}}^-,\tilde{d}^-)$ to $\tilde{\mathbf{h}}_{\alpha,p}^+ \in H_*(\tilde{\mathfrak{P}}^+,\tilde{d}^+)$.*

This theorem is an immediate extension of the theorem in [EGH] which states that for different choices of auxiliary data the Poisson algebras $H_*(\mathfrak{P}^-,d^-)$ and $H_*(\mathfrak{P}^+,d^+)$ with $d^\pm = \{\mathbf{h}^\pm,\cdot\}$ are isomorphic. In particular the extension in [F2] is about the coherent collection of sections $(s_j^\pm)$. Here one needs a notion of a collection of sections (s_j) in the tautological line bundles over all moduli spaces of holomorphic curves in the cylindrical cobordism interpolating between the auxiliary structures, which are *coherently connecting* the two coherent collections of sections $(s_j^\pm)$.

We want to point out the fact that the primary Poisson SFT homology algebra can be thought of as the space of functions on some abstract infinite-dimensional Poisson super-space. Indeed, consider the Poisson super-space $\mathbf{V}$ underlying the Poisson algebra $\mathfrak{P}$. Then the kernel $\ker(\{\mathbf{h},\cdot\})$ can be seen as the algebra of functions on the space $\mathcal{O}$ of orbits in $\mathbf{V}$ of the Hamiltonian $\mathbb{R}$-action given by $\mathbf{h}$, that is, the flow lines of the Hamiltonian vector field $X_\mathbf{h}$ associated to $\mathbf{h}$. Even in a finite dimensional setting the space $\mathcal{O}$ can be very wild. Anyhow the image $\text{im}(\{\mathbf{h},\cdot\})$ is an ideal of such algebra and hence identifies a sub-space of $\mathcal{O}$ given by all of those orbits $o \in \mathcal{O}$ at which, for any $f \in \mathfrak{P}$, $\{\mathbf{h},f\}|_o = 0$. But such orbits are simply the constant ones, where $X_\mathbf{h}$ vanishes. Hence the Poisson SFT-homology algebra $H_*(\mathfrak{P},\{\mathbf{h},\cdot\})$ can be regarded as the algebra of functions on $X_\mathbf{h}^{-1}(0)$, seen as a subspace of the space $\mathcal{O}$ of orbits of $\mathbf{h}$, endowed with a Poisson structure by singular, stationary reduction. In particular the descendant Hamiltonians $\mathbf{h}_{\alpha,j} := \frac{\partial\tilde{\mathbf{h}}}{\partial t^{\alpha,j}}\Big|_{t^{\alpha,j}=0,j>0} \in H_*(\mathfrak{P},\{\mathbf{h},\cdot\})$ are examples of functions on such space.

Finally we recall a result from [FR] that states that, besides commutativity, the SFT Hamiltonians satisfy analogues of the well-known string, dilaton and divisor equations of Gromov-Witten theory. Such equations hold, after passing to SFT homology, for any auxiliary choice used to define the Hamiltonians.

Theorem 1.7. *For any choice of differential forms and coherent sections the following* string, dilaton and divisor equations *hold* after *passing to SFT-homology*

$$\frac{\partial}{\partial t^{1,0}}\tilde{\mathbf{h}} = \int_V t \wedge t + \sum_k t^{\alpha,k+1}\frac{\partial}{\partial t^{\alpha,k}}\tilde{\mathbf{h}} \in H_*(\tilde{\mathfrak{P}}, \{\tilde{\mathbf{h}}, \cdot\}),$$

$$\frac{\partial}{\partial t^{1,1}}\tilde{\mathbf{h}} = \mathbf{D}_{\mathrm{Euler}}\,\tilde{\mathbf{h}} \in H_*(\tilde{\mathfrak{P}}, \{\tilde{\mathbf{h}}, \cdot\}),$$

$$\left(\frac{\partial}{\partial t^{2,0}} - z\frac{\partial}{\partial z}\right)\tilde{\mathbf{h}} = \int_V t \wedge t \wedge \theta_2 + \sum_k t^{\alpha,k+1}c_{2\alpha}^{\beta}\frac{\partial\tilde{\mathbf{h}}}{\partial t^{\beta,k}} \in H_*(\tilde{\mathfrak{P}}, \{\tilde{\mathbf{h}}, \cdot\}),$$

where $t^{1,k}$ is the t-variable associated to the k-th descendant of the unity class $1 \in H^*(V)$, $t^{2,k}$ is the one associated with $\theta_2 \in H^2(V)$ and z the corresponding Novikov ring variable, and $\mathbf{D}_{\mathrm{Euler}}$ is the linear differential operator

$$\mathbf{D}_{\mathrm{Euler}} := 2 - \sum_\gamma p_\gamma\frac{\partial}{\partial p_\gamma} - \sum_\gamma q_\gamma\frac{\partial}{\partial q_\gamma} - \sum_{\alpha,p} t^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p}}.$$

Remark 1.8. We end this introductory section with a short discussion of the analytical foundations of SFT. All the algebraic results we prove for holomorphic curves rely on the fact that all appearing moduli spaces are (weighted branched) manifolds with corners of dimension equal to the Fredholm index of the Cauchy-Riemann operator. In order to equip the zero set of the Cauchy-Riemann operator with nice manifold structures, one applies an infinite-dimensional version of the classical implicit function theorem, where the crucial step is to prove a transversality result for the Cauchy-Riemann operator. While it is well-known that transversality holds for a generic choice of almost complex structure as long as all holomorphic curves are simple, several problems appear when the curve is multiply-covered. Since this problem is already present in (symplectic) Gromov-Witten theory and Floer homology, very involved tools like virtual moduli cycles, Kuranishi structures and polyfolds were developed to solve the transversality problem for holomorphic curves in general. While the polyfold approach of Hofer, Wysocki and Zehnder, see [HWZ], seems to be the approach that solves all the challenges in the most satisfactory way (see also the survey [F3]), it is not yet fully completed. Since they promise to prove transversality for symplectic field theory and Gromov-Witten theory in one of their upcoming papers, we follow other papers in the field in proving everything up to transversality and state it nevertheless as a theorem. Since, in contrast to relative SFT and other papers about SFT using Morse-Bott techniques, we only need to add a transversality result for sections in our finite-dimensional tautological line bundles, our results are indeed rigorous when Hofer and his collaborators have completed their work.

$\triangle$

2. N-RECURSION FOR CONTACT HOMOLOGY

2.1. **Contact homology.** Contact homology is a reduced version of symplectic field theory that only uses curves with at most one positive puncture. In case the target manifold V is contact, the maximum principle for holomorphic curves in symplectizations forbids that a SFT-holomorphic curve with s^+ positive punctures can degenerate to a multi-floor curve where any of the components has more than s^+ positive punctures. The absence of local maxima further implies that any curve with no positive punctures must be constant. In particular this means that we can study moduli spaces of SFT-curves with only one positive puncture and be safe that the boundary only involves moduli spaces of the same type. Besides, in such moduli spaces nodal degeneration only involve the appearance of constant bubbles (when different marked points come together).

The algebraic structure we obtain is the linear part in the p-variables of the one described for full SFT. In particular we get a complex formed by the graded commutative algebra $\mathcal{A}$ generated by the variables q_γ over the power series in the variables $t^\alpha = t^{\alpha,0}$ with coefficients in the Novikov ring of variables z_k (i.e. the evaluation at $p = 0$ of the Poisson algebra $\mathfrak{P}$), and differential given by the (odd) vector field

$$X = \sum_\gamma X^\gamma \frac{\partial}{\partial q_\gamma} = \sum_\gamma \kappa_\gamma \left. \frac{\partial \mathbf{h}}{\partial p_\gamma} \right|_{p=0} \frac{\partial}{\partial q_\gamma}.$$

The master equation $\{\mathbf{h}, \mathbf{h}\} = 0$ reduces to $[X, X] = 0$ (the square bracket here stands for Lie bracket on vector fields), the resulting homology will be denoted by $CH(\mathcal{A}; X)$ (or simply $CH(V)$ when there is no danger of confusion) and the system of commuting (on SFT-homology) descendant Hamiltonians $\mathbf{h}_{\alpha,i} = \left. \frac{\partial \bar{\mathbf{h}}}{\partial t^{\alpha,i}} \right|_{t^{\beta,j}=0, j>0}$, $(\alpha, i) \in \{1, \ldots, N\} \times \mathbb{N}$ induce a system of Lie-commuting vector fields on contact homology

$$X_{\alpha,i} = \sum_\gamma \kappa_\gamma \left. \frac{\partial \mathbf{h}_{\alpha,i}}{\partial p_\gamma} \right|_{p=0} \frac{\partial}{\partial q_\gamma} : CH(V) \to CH(V).$$

2.2. **The non-equivariant differential revisited.** Consider now a moduli spaces of punctured S^1-parametrized cylinders with marked points. We start with the fully parametrized space $\mathcal{M}_{r,A}^{S^1,0}(\gamma_0, \gamma_\infty, \Gamma^-)$ consisting of tuples $(u, (z_k^-), (z_i))$, where $(z_k^-), (z_i)$ are two disjoint ordered sets of points on $\mathbb{CP}^1 - (\{0, \infty\}) = S^1 \times \mathbb{R}$ (namely negative punctures with asymptotic markers, and r additional marked points). The map $u : \dot{S} \to \mathbb{R} \times V$ from the punctured Riemann surface $\dot{S} = \mathbb{CP}^1 - (\{0, \infty\} \cup \{(z_k^-)\})$ is required to satisfy the Cauchy-Riemann equation

$$\bar{\partial}_{\underline{J}} u = du + \underline{J}(u) \cdot du \cdot i = 0$$

with the complex structure i on $\mathbb{CP}^1$. Assuming we have chosen cylindrical coordinates $\psi_k^- : \mathbb{R}^- \times S^1 \to \dot{S}$ around each puncture z_k^-, in the sense that $\psi_k^-(-\infty, t) = z_k^-$, the map u is additionally required to show for all $k = 1, \ldots, n^-$ the asymptotic behaviour

$$\lim_{s \to -\infty} (u \circ \psi_k^-)(s, t + t_0) = (-\infty, \gamma_k^-(T_k^- t))$$

with some $t_0 \in S^1$ and the orbits $\gamma_k^- \in \Gamma^-$, where $T_k^- > 0$ denotes period of γ_k^-, and analogous asymptotic behaviour at 0 and ∞ for $s \to +\infty$ and $s \to -\infty$ respectively, for orbits γ_0 and γ_∞ and with respect to the natural coordinates on $S^1 \times \mathbb{R}$. We assign to each curve an absolute homology class A employing as usual a choice of spanning surfaces. In order to obtain the S^1-parametrized space $\mathcal{M}_{r,A}^{S^1}(\gamma_0, \gamma_\infty, \Gamma^-)$

we only divide out the $\mathbb{R}$-component of the $S^1 \times \mathbb{R}$ group of automorphisms of the cylinder $\mathbb{CP}^1 - (\{0, \infty\})$ and, as usual, the $\mathbb{R}$-action coming from the cylindrical target $V \times \mathbb{R}$ as well.

This moduli space can be compactified to $\overline{\mathcal{M}}^{S^1} = \overline{\mathcal{M}}^{S^1}_{r,A}(\gamma_0, \gamma_\infty, \Gamma^-)$ by adding moduli spaces of S^1-parametrized multi-floor curves with ghost bubbles, where the puncture at 0 is always the positive puncture of the top floor, the puncture at ∞ can be on any floor and the S^1-parametrization is remembered when going through connecting punctures as explained in [BEHWZ] (compactification of the space of curves with *decorations*). More explicitly, in such compactification, a n-floor curve with the ∞-puncture on its k-th component from the top, has the upper k components that are S^1-parametrized curves where, at each connecting puncture, the S^1-coordinates of different components match, while the lower $(n-k)$ are non-S^1-parametrized. Also, both type of components possibly have stable constant bubbles.

The space $\overline{\mathcal{M}}^{S^1}$ carries, besides the usual evaluation maps at marked points, orbits and asymptotic markers, extra evaluation maps at the punctures at 0 and ∞ to the corresponding target simple Reeb orbits given by the special S^1-coordinate on the curve,

$$\mathrm{ev}_{+\infty,0} : \overline{\mathcal{M}}^{S^1} \to \bar{\gamma}_0 \simeq S^1$$
$$\mathrm{ev}_{-\infty,\infty} : \overline{\mathcal{M}}^{S^1} \to \bar{\gamma}_\infty \simeq S^1.$$

We form the space $\overline{\mathcal{M}}^{S^1}_{r,n^-,A}(\gamma_0, \gamma_\infty)$ by taking the union over Γ^- of all the spaces $\overline{\mathcal{M}}^{S^1}_{r,A}(\gamma_0, \gamma_\infty, \Gamma^-)$ with $|\Gamma^-| = n^-$.

This moduli space was actually already introduced in [BO] to define the non-S^1-equivariant linearized contact homology differential. We will proceed in a similar way defining a $(1,1)$-tensor $N(t^\alpha)$ depending on parameters $t^1, \ldots, t^N$, on the super-space $\mathbf{Q}$ underlying the algebra $\mathcal{A}$. A point $q \in \mathbf{Q}$ is a cohomology class $q = \sum_{\gamma \in \mathcal{P}} \frac{1}{\kappa_\gamma} q_\gamma [\gamma]$ on $\mathcal{P}$, with the notations of section 1. We will write q^γ instead of q_γ to be coherent with the notion that q_γ is treated here as a coordinate for the space $\mathbf{Q}$, while we will treat the t-variables as parameters on which the functions on $\mathbf{Q}$ can depend. Using such coordinates we define

$$N = N_{\gamma_1}^{\gamma_2}(t, q) \; dq^{\gamma_1} \otimes \frac{\partial}{\partial q^{\gamma_2}}$$

where we sum over repeated indices, and

$$N_{\gamma_1}^{\gamma_2}(t, q) = \sum \frac{1}{r! n^-! \kappa_{\gamma_1}} \int_{\overline{\mathcal{M}}^{S^1}_{r,n^-,A}(\gamma_1, \gamma_2)} \bigwedge_{i=1}^{r} \mathrm{ev}_i^* \, t \bigwedge_{j=1}^{n^-} (ev_{-,j}^* q \wedge \mathrm{ev}_{-\infty,j}^* \, d\phi_{\bar{\gamma}_j^-}) \wedge$$
$$\wedge \, \mathrm{ev}_{+\infty,0}^* \, d\phi_{\bar{\gamma}_1} \wedge \mathrm{ev}_{-\infty,\infty}^* \, d\phi_{\bar{\gamma}_2}$$

With the usual grading of the SFT variables, and assigning degree 0 to the exterior differential d on the superspace $\mathbf{V}$, from the index formula for the dimension of the moduli space of SFT-curves, we deduce

$$|N| = -2.$$

Notice that the no-descendant (or primary) contact homology differential X (still parametrized by the primary variables t^α) induces a differential $\mathcal{L}_X$ (Lie derivative along the vector field X) on the space of (k,l)-tensor fields $\mathcal{T}^{(k,l)}\mathbf{Q}$ (again with

parameters t^α) on the super-space $\mathbf{Q}$. The resulting homology, which we denote $CH(\mathcal{T}^{(k,l)}\mathbf{Q}; \mathcal{L}_X)$, is a module over $CH(\mathcal{A}; X) = CH(\mathcal{T}^{(0,0)}\mathbf{Q}; \mathcal{L}_X)$ and is an invariant of the contact structure on V, as it can easily be proved with the same procedure as for $CH(\mathcal{A}; X)$. In particular, for two different choices of contact form $\lambda^\pm$, cylindrical almost complex structure $\underline{J}^\pm$, representatives for the classes $[\theta_\alpha] \in H^*(V)$ and $[d\phi_{\bar\gamma}] \in H^*(S^1)$, abstract polyfold perturbations and sequences of coherent collections of sections $(s_j^\pm)$, there exist an isomorphism

$$d\varphi^\pm : CH(\mathcal{T}^{(k,l)}\mathbf{Q}^+, \mathcal{L}_{X^+}) \to CH(\mathcal{T}^{(k,l)}\mathbf{Q}^-, \mathcal{L}_{X^-})$$

which is simply the lift to the tensor algebra of the isomorphism

$$\varphi^\pm : CH(\mathcal{A}^+; X^+) \to CH(\mathcal{A}^-; X^-),$$

constructed in [EGH] by studying curves in the cobordims $W = \overrightarrow{V^+V^-}$ interpolating between the two different choices (see also the discussion on invariance for satellites there).

Theorem 2.1.

$$\mathcal{L}_X N = 0$$

and, denoting by $N^\pm$ the two $(1,1)$-tensors resulting from two different choices of contact form $\lambda^\pm$, cylindrical almost complex structure $\underline{J}^\pm$, representatives for the classes $[\theta_\alpha] \in H^*(V)$ and $[d\phi_{\bar\gamma}] \in H^*(S^1)$, abstract polyfold perturbations and sequences of coherent collections of sections $(s_j^\pm)$,

$$d\varphi^\pm : CH(\mathcal{T}^{(1,1)}\mathbf{Q}^+, \mathcal{L}_{X^+}) \;\to\; CH(\mathcal{T}^{(1,1)}\mathbf{Q}^-, \mathcal{L}_{X^-})$$
$$N^+ \qquad\qquad \mapsto \qquad\qquad N^-$$

so that $N \in CH(\mathcal{T}^{(1,1)}\mathbf{Q}, \mathcal{L}_X)$ is an invariant of the contact structure on V.

Proof. For the proof of the equation $\mathcal{L}_X N = 0$ we need to study the boundary of the moduli spaces involved in the definition of N, i.e. moduli spaces of punctured $\mathbb{CP}^1$ with a marked $\mathbb{R}^+$ line connecting two punctures 0 and ∞ mapped to orbits γ_1 and γ_2, at which we pull back 1-forms from the underlying simple orbits. In the picture below we represent such moduli by drawing the corresponding generic element (the curve with the marked red $\mathbb{R}^+$ line) and we rapresent the constraining of the endpoints via 1-forms by the small triangles (a triangle pointing towards an orbit γ means that the red line is S^1-constrained at that orbit by integrating the pull-back of the form $d\phi_{\bar\gamma}$). The three kind of curve degenerations that form this (codimension 1) boundary are drawn as the corresponding 2-floor curves with constrained or unconstrained red lines (here we use the fact that, when a curve splits at a puncture γ through which the $\mathbb{R}^+$ line passes, we need to pull back a representative of the diagonal class in $H^*(\gamma \times \gamma)$, which we express as $\frac{1}{\kappa_\gamma}(d\phi_{\bar\gamma} \otimes 1 + 1 \otimes d\phi_{\bar\gamma})$).
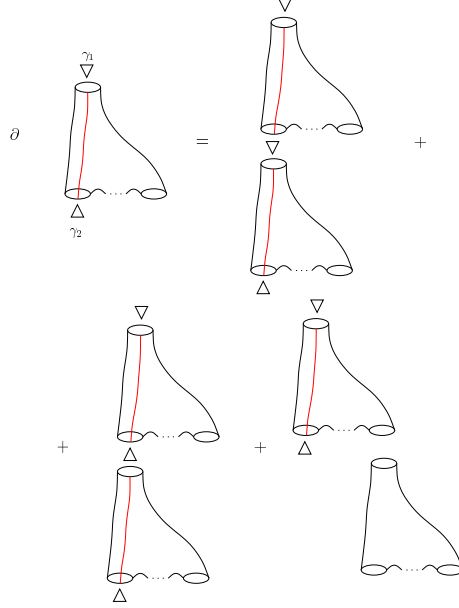
All we need to notice, at this point, is that, taking orientation into account for the right signs, the three terms on the righ-hand side of the equation represented in the picture exactly correspond to the three summands in the coordinate expression of the Lie derivative $\mathcal{L}_X N$, which hence vanishes by Stokes theorem applied to the boundary of our moduli space.

For the second part of theorem, about invariance with respect to auxiliary choices in the definition of $\mathcal{A}$, X and N a similar approach is needed, where we study the boundary of the moduli spaces of the same type of curves, but this time in the cobordism interpolating between two different choices of auxiliary data. Drawing again the same kind of pictures (only remembering, as explained in [BEHWZ], that the boundary of moduli spaces of connected curves in the cobordism is formed by 2-floor curves in which one of the floors is a connected curve in the cylindrical manifold over one of the boundaries and the other is a possibly disconnected curve in the cobordism). Algebraically this gives precisely the transformation rule for N described in the statement with respect to the lift $d\varphi^\pm$ of the isomorphism $\varphi^\pm$ to the homology tensor algebras of $\mathbf{Q}^\pm$. $\qquad\square$

Corollary 2.2. *For any $\alpha = 1, \ldots, N$ and any $i = 0, 1, 2, \ldots$*

$$\mathcal{L}_{X_{\alpha,i}} N = 0 \ \in CH(\mathcal{T}^{(1,1)}\mathbf{Q}, \mathcal{L}_X)$$

Proof. Simply expand the equation $\mathcal{L}_X N = 0$ in powers of the t-variables and consider the linear terms. $\qquad\square$

Example 2.3. Consider the case $V = S^1$ with $t = t^1\theta_1 + \tau^1\Theta_1$, $\theta_1 = 1$ and $\Theta_1 = d\varphi$ where φ is the angular coordinate on S^1. It is easy to compute $\bar{N} := N|_{\tau^1=0}$. Writing just k for the index k_γ associated to the k-th multiple of the orbit $\gamma = V$, from the dimension formula for the moduli space of SFT-curves and an easy curve counting we immediately see that

$$\bar{N}_k^l = \frac{l-k}{k}q^{l-k}, \qquad l > k$$
$$\bar{N}_k^l = 0, \qquad\qquad l \le k$$

$\triangle$

2.3. **Vanishing of Nijenhuis torsion of N.**

Theorem 2.4.
$$\left(N_\gamma^{\gamma_1} \left(\frac{\partial N_{\gamma_2}^\gamma}{\partial q^{\gamma_3}} - \frac{\partial N_{\gamma_3}^\gamma}{\partial q_2^\gamma} \right) - \frac{\partial N_{\gamma_2}^{\gamma_1}}{\partial q^\gamma} N_{\gamma_3}^\gamma + \frac{\partial N_{\gamma_3}^{\gamma_1}}{\partial q^\gamma} N_{\gamma_2}^\gamma \right) \frac{\partial}{\partial q^{\gamma_1}} \otimes dq^{\gamma_2} \otimes dq^{\gamma_3} = 0$$
$$\in CH(\mathcal{T}^{(1,2)}\mathbf{Q}, \mathcal{L}_X)$$

Proof. For the proof we need to study the boundary of the moduli spaces of contact homology curves with three special punctures, the positive one and two of the negative ones. Without loss of generality, in the interior of such moduli spaces, the positive puncture can be the image of $0 \in \mathbb{CP}^1$, and the two negative ones of $i \in \mathbb{CP}^1$ and $\infty \in \mathbb{CP}^1$. Then the real half-line $\mathbb{R}^+$ and the half-circle $\{|z - i/2|^2 = 1/4, \ \mathrm{Re}(z) \geq 0\} \subset \mathbb{CP}^1$ determine asymptotic directions at the three points (the one at 0 pointing along $\mathbb{R}^+$, the one at i and ∞ pointing towards 0 along the half-circle and $\mathbb{R}^+$ respectively). These will be constrained to asymptotic markers at the corresponding Reeb orbits by pulling back classes $d\phi_{\gamma_1} \otimes d\phi_{\gamma_1}$, $d\phi_{\gamma_2}$ and $d\phi_{\gamma_3}$ at the orbits γ_1, γ_2 and γ_3 respectively.
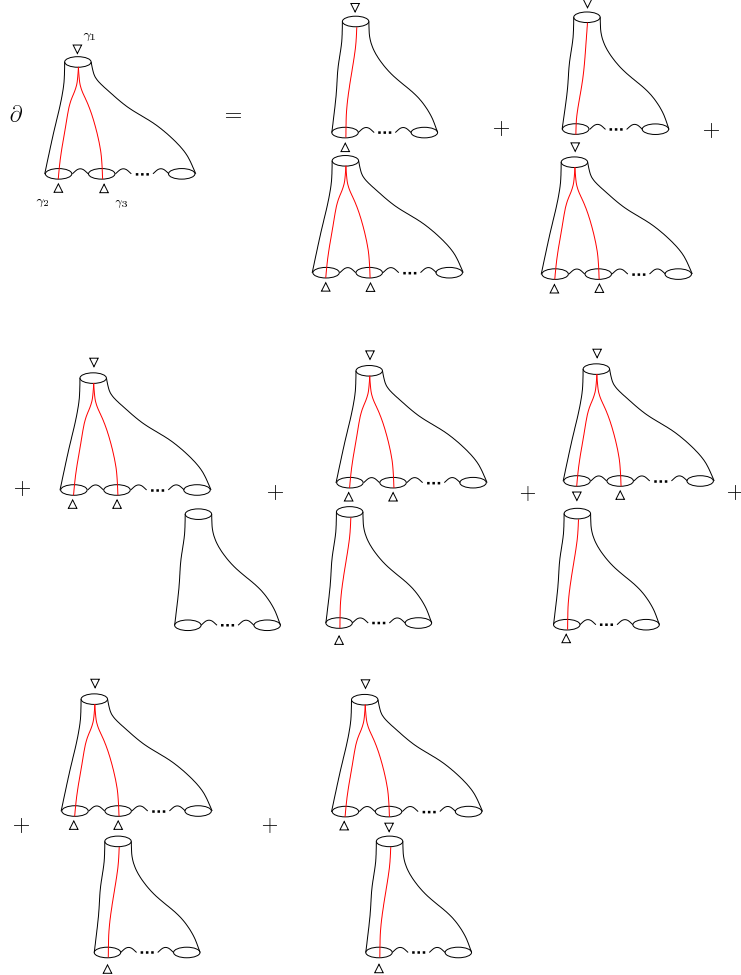
The above picture, with the usual notations, represents the possible codimension-1 boundary degenerations in such moduli spaces. The two red lines, with common tangent direction at the positive puncture, represent the half-circle and the $\mathbb{R}^+$ line.

The second, third, fourth and sixth term of the right-side form the Lie derivative along X of the $(1,2)$-tensor whose correlator counts the holomorphic curves with three special punctures described above, and hence disappear when taking homology with respect to $\mathcal{L}_X$. The first term spells out as $N_\gamma^{\gamma_1} \left(\frac{\partial N_{\gamma_2}^\gamma}{\partial q^{\gamma_3}} - \frac{\partial N_{\gamma_3}^\gamma}{\partial q_2^\gamma} \right)$. Indeed the curves represented in the lower floor of the first summand have two red lines on them, whose tangents at the positive puncture match but are not constrained to meet any asymptotic marker on the Reeb orbit. The matching condition can be once more expressed by pulling back to the moduli space the form $\frac{1}{\kappa_\gamma}(d\phi_{\bar\gamma} \otimes 1 - 1 \otimes d\phi_{\bar\gamma}))$ representing the anti-diagonal class in $H^*(\gamma \times \gamma)$ (the class of the diagonal in $S^1 \times (-S^1)$, where the minus in the second factor denotes reversed orientation), which immediately gives the factor $\left(\frac{\partial N_{\gamma_2}^\gamma}{\partial q^{\gamma_3}} - \frac{\partial N_{\gamma_3}^\gamma}{\partial q_2^\gamma} \right)$.

The remaining summands, the fifth and the seventh, give the remaining part of the master equation.

$\square$

Corollary 2.5. *For any $Y \in CH(\mathcal{T}^{(1,0)}\mathbf{Q}, \mathcal{L}_X)$,*

$$\mathcal{L}_Y N = 0 \ \in CH(\mathcal{T}^{(1,1)}\mathbf{Q}, \mathcal{L}_X) \quad \Rightarrow \quad \mathcal{L}_{N(Y)} N = 0 \ \in CH(\mathcal{T}^{(1,1)}\mathbf{Q}, \mathcal{L}_X)$$

Proof. Simply spell out in components the difference $\mathcal{L}_Y N - \mathcal{L}_{N(Y)} N$ to see that it is proportional to the left-hand side of the master equation for N. $\square$

2.4. **Descendant vector fields and N-recursion.** The following result shows how the non-equivariant Nijenhuis endomorphism N is related to the geometry of gravitational descendants and the combined knowledge of the primary vector fields $X_{\alpha,0} \in CH(\mathcal{T}^{(1,0)}\mathbf{Q}, \mathcal{L}_X)$ and of the endomorphism $N \in CH(\mathcal{T}^{(1,1)}\mathbf{Q}, \mathcal{L}_X)$ allows for completely recovering all of the descendant vector fields $X_{\alpha,i} \in CH(\mathcal{T}^{(1,0)}\mathbf{Q}, \mathcal{L}_X)$, $i > 0$.

Theorem 2.6.

$$X_{\alpha,n} = N(X_{\alpha,n-1}) + C_{\alpha,n-1}^\mu X_{\mu,0} \ \in CH(\mathcal{T}^{(1,0)}\mathbf{Q}, \mathcal{L}_X)$$

where

$$C_{\alpha,n}^\mu = C_{\alpha,n}^\mu(t) = \frac{\partial^2}{\partial t^\alpha \partial t^\nu} \int_V \frac{t^{\wedge(n+3)}}{(n+3)!} \ \eta^{\nu\mu}$$

Proof. Once more we need to study the codimension-1 boundary of a moduli space of curves. In this case we consider contact homology curves with three special points: the positive puncture at the orbit γ, a marked point at which we pull back the unity class $1 \in H^*(V)$ and another marked point carrying the n-nth descendant of the class $\theta_\alpha \in H^*(V)$ (and no other point carries gravitational descendants). Mapping these three points to $\{0, 1, \infty\} \in \mathbb{CP}^1$ we obtain an asymptotic direction at the positive puncture given by the $\mathbb{R}^+$-line and we constrain such direction as usual via the asymptotic marker at the corrisponding positive Reeb orbit.
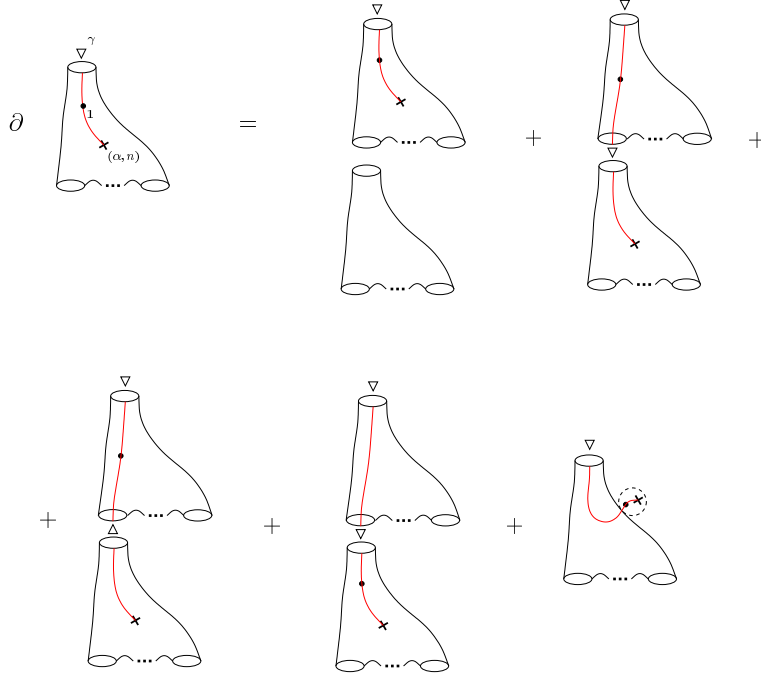
In the usual way the above picture shows the different types of codimension-1 boundary degeneration for such moduli space. Notice that, since the two special marked points are constrained to a line which is in turn S^1-constrained at the Reeb orbit, a special kind of codimension-1 phenomenon appears, which is not anymore a 2-floor curve, but is instead a 1-floor curve with a constant sphere-bubble carrying the two special marked points (which corresponds to the limit where the point carrying the class $1 \in H^*(V)$ moves along the $\mathbb{R}^+$-line to reach the other marked point carrying the descendant), represented as the last term in the right hand side in the picture.

Now we notice that the first and fourth terms in the right-hand side correspond to the Lie derivative along X of a vector field on $\mathbf{Q}$ whose component along $\frac{\partial}{\partial q^\gamma}$ are given by the correlator counting the curves described above. Notice further that the factor corresponding to top floor in the second summand is zero unless the curve is a constant cylinder, because the marked point carrying the class $1 \in H^*(V)$ is always unconstrained along the red line and the only way to achieve a zero-dimensional moduli space is by quotienting out the vertical symmetry in constant cylinders over the Reeb orbit γ . Taking homology, what is left can be spelled out as

$$\left(X_{\alpha,n}^{\gamma_1} \delta_{\gamma_1}^\gamma - \frac{\partial X_{\alpha,n}^{\gamma_1}}{\partial t^1} N_{\gamma_1}^\gamma - \frac{\partial C_{\alpha,n}^\mu}{\partial t^1} X_{\mu,0}^\gamma \right) \frac{\partial}{\partial q^\gamma} = 0 \ \in CH(\mathcal{T}^{(1,0)}\mathbf{Q}, \mathcal{L}_X)$$

where $C_{\alpha,n}^\mu$ is the term accounting for the constant bubbles with one psi-class to the power n. Such term is easily calculated from the well known fact that, on the Deligne-Mumford space of genus 0 curves with r marked points,

$$\int_{\overline{\mathcal{M}}_{0,r}} \psi_i^n = \left\{ \begin{array}{l} 1, \ r = n+3 \\ 0, \ r \neq n+3 \end{array} \right.$$

Finally we need to use the string equation of Theorem 1.7 on the above equation to obtain the statement. $\square$

Corollary 2.7.

$$X_{\alpha,n} = \sum_{k=0}^{n} C_{\alpha,n-k-1}^{\mu} \, N^k(X_{\mu,0}) \ \in CH(\mathcal{T}^{(1,0)}\mathbf{Q}, \mathcal{L}_X)$$

where

$$C_{\alpha,n}^{\mu} = C_{\alpha,n}^{\mu}(t) = \frac{\partial^2}{\partial t^\alpha \partial t^\nu} \int_V \frac{t^{\wedge(n+3)}}{(n+3)!} \, \eta^{\nu\mu}$$

Proof. Just apply Theorem 2.6 n times to $X_{\alpha,n}$. $\qquad\qquad\square$

Naturally the above theorem and corollary hold for any choice of auxiliary data given the completely covariant behaviour of the equations. In particular the above corollary shows how the descendant vector fields $X_{\alpha,n}$ are expressed in closed form in terms of the primary vector fields $X_{\alpha,0}$ and the endomorphism N.

Example 2.8. Consider again the example of $V = S^1$. In this case we have $\bar{X} := X|_{\tau^1=0} = 0$ and $\bar{X}_{1,n} := \frac{\partial X}{\partial \tau^{1,n}}\big|_{\tau=0}$, with $\bar{X}_{1,0}^k = kq^k$, where as before we write k for the index $k\gamma$ associated to the k-th multiple of the orbit $\gamma = V$.

Applying the above Theorem 2.6 we obtain

$$\bar{X}_{1,1}^l = \bar{X}_{1,0}^k \bar{N}_k^l + \bar{C}_{1,0}^1 \bar{X}_{1,0}^j$$

. Here $\bar{C}_{1,n}^1 = C_{1,n}^1|_{\tau^1=0} = \frac{(t^1)^{n+1}}{(n+1)!}$, hence we obtain

$$\bar{X}_{1,1}^l = l\,t^1 q^l + \sum_{0<k<l} (l-k)q^k q^{l-k} =$$

$$= l\,t^1 q^l + \frac{1}{2}\left(\sum_{0<k<l} (l-k)q^k q^{l-k} + \sum_{0<k'<l} k' q^{-k'+l} q^{k'} \right) =$$

$$= l\,t^1 q^l + \frac{l}{2} \sum_{0<k<l} q^k q^{l-k}$$

and, with the same procedure we obtain, applying N n times and denoting $q^0 := t^1$,

$$\bar{X}_{1,n}^l = \frac{l}{(n-1)!} \sum_{\substack{k_1,\ldots,k_n \geq 0 \\ k_1+\ldots+k_n=l}} q^{k_1} \ldots q^{k_n}$$

(one needs to use the following trick

$$\sum_{\substack{k_1,\ldots,k_n \geq 0 \\ k_1+\ldots+k_n \leq l}} (l - k_1 - \ldots - k_n)\, q^{k_1} \ldots q^{k_n} q^{l-k_1-\ldots-k_n} =$$

$$= \frac{1}{n}\left(\sum_{\substack{k_1',k_2,\ldots,k_n \geq 0 \\ k_1'+k_2+\ldots+k_n \leq l}} k_1' q^{l-k_1'-k_2-\ldots-k_n} + \right.$$

$$\left. + \ldots + \sum_{\substack{k_1,\ldots,k_{n-1},k_n' \geq 0 \\ k_1+\ldots+k_{n-1}+k_n' \leq l}} k_n' q^{l-k_1-\ldots-k_{n-1}-k_n'} \right) =$$

$$\frac{l}{n} \sum_{\substack{k_1,\ldots,k_n \geq 0 \\ k_1+\ldots+k_n \leq l}} q^{k_1} \ldots q^{k_n} q^{l-k_1-\ldots-k_n}$$

to take the numerical coefficient out of the sum). $\qquad\qquad\triangle$

3. ω-RECURSION IN RATIONAL SFT

An approach similar to the one we used above for contact homology also works in the case of full rational SFT, the main difference coming from the presence of non-constant nodal curves which is very naturally incorporated in the algebraic formalism of Lie derivatives and tensor fields by trading the Nijenhuis operator N for a bivector ω well defined on SFT homology.

3.1. The ω bivector in rational SFT.

For any target stable Hamiltonian structure $(V, \Omega, \lambda, R, \underline{J})$, consider the following moduli spaces of punctured S^1-parametrized cylinders with marked points. We start with the fully parametrized space

$$\mathcal{M}_{r,A}^{S^1;0}((\gamma_0, \pm), (\gamma_\infty, \pm), \Gamma^+, \Gamma^-))$$

consisting of tuples $(u, (z_k^\pm), (z_i))$, where $(z_k^+), (z_j^-), (z_i)$ are three disjoint ordered sets of points on $\mathbb{CP}^1 - (\{0, \infty\}) = S^1 \times \mathbb{R}$ (positive and negative punctures, and r additional marked points). The map $u : \dot{S} \to \mathbb{R} \times V$ from the punctured Riemann surface $\dot{S} = \mathbb{CP}^1 - (\{0, \infty\} \cup \{(z_k^+)\} \cup \{(z_k^+)\})$ is required to satisfy the Cauchy-Riemann equation

$$\bar{\partial}_{\underline{J}} u = du + \underline{J}(u) \cdot du \cdot i = 0$$

with respect to the complex structure i on $\mathbb{CP}^1$. Assuming we have chosen cylindrical coordinates $\psi_k^\pm : \mathbb{R}^\pm \times S^1 \to \dot{S}$ around each puncture $z_k^\pm$, in the sense that $\psi_k^\pm(\pm\infty, t) = z_k^\pm$, the map u is additionally required to show for all $k = 1, ..., n^\pm$ the asymptotic behaviour

$$\lim_{s \to \pm\infty} (u \circ \psi_k^\pm)(s, t + t_0) = (\pm\infty, \gamma_k^\pm(T_k^\pm t))$$

with some $t_0 \in S^1$ and the orbits $\gamma_k^\pm \in \Gamma^\pm$, where $T_k^\pm > 0$ denotes period of $\gamma_k^\pm$, and analogous asymptotic behaviour at 0 and ∞ for $s \to \pm\infty$ (the signs here correpond to the signs in $(\gamma_0, \pm), (\gamma_\infty, \pm)$ in the notation for the moduli space) for orbits γ_0 and γ_∞ with respect to the natural coordinates on $S^1 \times \mathbb{R}$. We assign to each curve an absolute homology class A employing as usual a choice of spanning surfaces. In order to obtain the S^1-parametrized space $\mathcal{M}_{r,A}^{S^1}((\gamma_0, \pm), (\gamma_\infty, \pm), \Gamma^+, \Gamma^-)$ we only divide out the $\mathbb{R}$-component of the $S^1 \times \mathbb{R}$ group of automorphisms of the cylinder $\mathbb{CP}^1 - (\{0, \infty\}$ and, as usual, the $\mathbb{R}$-action coming from the cylindrical target $V \times \mathbb{R}$ as well.

The compactification $\overline{\mathcal{M}}^{S^1} = \overline{\mathcal{M}}_{r,A}^{S^1}((\gamma_0, \pm), (\gamma_\infty, \pm), \Gamma^+, \Gamma^-)$ is obtained as usual by adding multifloor S^1-parametrized curves. In genus zero each floor has only one non-trivial connected component, all the others being trivial cylinders over Reeb orbits (possibly S^1-parametrized). If the 0 and ∞ punctures determining the S^1-parametrization appear on the k-th and l-th floor of a n-floor curve, it means that all non-trivial curves appearing on the m-th floor are S^1-parametrized when $k \le m \le l$ and ordinary unparametrized curves when $m < k$ or $m > l$. Nodal curves must also be added to the picture, as usual, with the only remark that, when a node separates the 0 and ∞ punctures on a given floor, each component carries his own S^1-parametrization with respect to the node and the 0 or ∞ puncture respectively.

As in the contact homology case, the space $\overline{\mathcal{M}}^{S^1}$ carries, besides the usual evaluation maps at marked points, orbits and asymptotic markers, extra evaluation maps at the punctures at 0 and ∞ to the corresponding target simple Reeb orbits given by the special S^1-coordinate on the curve,

$$\mathrm{ev}_{\pm\infty,0} : \overline{\mathcal{M}}^{S^1} \to \bar{\gamma}_0 \simeq S^1$$

$$\mathrm{ev}_{\pm\infty,\infty} : \overline{\mathcal{M}}^{S^1} \to \bar{\gamma}_\infty \simeq S^1.$$

We form the space $\overline{\mathcal{M}}^{S^1}_{r,n^+,n^-,A}((\gamma_0,\pm),(\gamma_\infty,\pm))$ by taking the union over Γ^+ and Γ^- of all the spaces $\overline{\mathcal{M}}^{S^1}_{r,A}((\gamma_0,\pm),(\gamma_\infty,\pm),\Gamma^+,\Gamma^-)$ with $|\Gamma^-| = n^+$ and $|\Gamma^-| = n^-$.

We will now define a $(1,1)$-tensor on the Poisson super-space $\mathbf{V}_0$ underlying the Poisson subalgebra $\mathfrak{P}_0 \subset \mathfrak{P}$ generated by p and q-variables and even t-variables only. In other words, if $H^*(V) = H^*_{\mathrm{even}}(V) \oplus H^*_{\mathrm{odd}}(V)$, with $H^*_{\mathrm{even}}(V) = <\theta_1,\dots,\theta_N>$ and $H^*_{\mathrm{even}}(V) = <\Theta_1,\dots,\Theta_L>$, we denote by t^α the (even) formal variable associated to the class θ_α, $\alpha = 1,\dots,N$, and by $\tau^{\check\alpha}$ the (odd) formal variable associated to $\Theta_{\check\alpha}$, $\check\alpha = 1,\dots,L$. Then $\mathfrak{P}_0 = \mathfrak{P}|_{\tau=0}$. Correspondingly we define $\mathbf{h}^0 := \mathbf{h}|_{\tau=0}$ and $\mathbf{h}_{\check\alpha,n} := \frac{\partial \mathbf{h}}{\partial \tau^{\check\alpha,n}}|_{\tau=0}$. Notice that the Hamiltonians $\mathbf{h}_{\check\alpha,n}$ are always even elements in $\mathfrak{P}_0$.

We will denote globally by v^A any of the coordinates t^α, p^a or q^a (again, to avoid confusion, we have raised the indices of p and q variables, coherently with their interpretation as coordinates for $\mathbf{V}_0$). We will always use lower case roman indices (e.g. v^a) to refer indistinctly to a p or q variable, greek indices (e.g. v^α) for t variables and checked greek indices (e.g. $v^{\check\alpha}$) for τ-variables. Also, for convenience, we let $v^{(\gamma,+)} := p^\gamma$ and $v^{(\gamma,-)} := q^\gamma$, so that the roman upper case indices A, B, etc. can take the values α, β, etc. when the corresponding variable is a t-variable, or the values $(\gamma_1,\pm)$, $(\gamma_2,\pm)$, etc. or again simply a and b when the corresponding variable is a p or q-variable. Notice also that, as opposed to what we did for the space $\mathbf{Q}$, the t-variables are treated here as genuine coordinates and not as parameters. In particular, the Poincaré metric η will split into two blocks (one the transpose of the other) always pairing even with odd cohomology classes. We denote the matrix corresponding to each of such blocks by $\eta_{\alpha\check\alpha} = \eta(\theta_\alpha,\Theta_{\check\alpha})$.

Using such coordinates we define the (graded) symmetric bivector

$$\omega = \omega^{AB} \frac{\partial}{\partial v^A} \otimes \frac{\partial}{\partial v^B}$$

where we sum over repeated indices, with

$$\omega^{(\gamma_1,\pm)(\gamma_2,\pm)} = \sum \frac{1}{r!n^-!} \int_{\overline{\mathcal{M}}^{S^1}_{r,n^+,n^-,A}((\gamma_1,\pm),(\gamma_2,\pm))} \bigwedge_{i=1}^r \mathrm{ev}_i^* t \bigwedge_{j=1}^{n^+} (ev^*_{+,j} p \wedge \mathrm{ev}^*_{+\infty,j} \, d\phi_{\bar\gamma_j^+})$$

$$\bigwedge_{j=1}^{n^-} (ev^*_{-,j} q \wedge \mathrm{ev}^*_{-\infty,j} \, d\phi_{\bar\gamma_j^-}) \wedge (\mathrm{ev}^*_{\pm\infty,0} \, d\phi_{\bar\gamma_1}) \wedge (\mathrm{ev}^*_{\mp\infty,\infty} \, d\phi_{\bar\gamma_2})$$

and

$$\omega^{a\alpha} = \Pi^{ab} \eta^{\alpha\check\mu} \frac{\partial \mathbf{h}_{\check\mu,n}}{\partial v^b}\bigg|_{\tau=0} = \omega^{\alpha a}$$

and zero otherwise.

As in contact homology, from the index formula for the virtual dimension of the moduli space of SFT curves, we have

$$|\omega| = -2.$$

Algebraically, the SFT differential $d_0 : \mathfrak{P}_0 \to \mathfrak{P}_0$, defined as the vector field $X^0 = X_{\mathbf{h}^0} = \{\mathbf{h}^0, \cdot\} : \mathfrak{P}_0 \to \mathfrak{P}_0$ induces a differential $\mathcal{L}_{X^0}$ on the space of (k,l)-tensor fields $\mathcal{T}^{(k,l)}\mathbf{V}_0$ on the Poisson super-space $\mathbf{V}_0$ underlying $\mathfrak{P}_0$. The resulting homology, which we denote by $H_*(\mathcal{T}^{(k,l)}\mathbf{V}_0; \mathcal{L}_{X^0})$, is a module over $H_*(\mathfrak{P}_0, d_0) = H_*(\mathcal{T}^{(0,0)}\mathbf{V}_0; \mathcal{L}_{X^0})$ and is an invariant of the stable Hamiltonian structure on V. In particular, for two different choices of form $\lambda^\pm$, cylindrical almost complex structure $\underline{J}^\pm$, representatives for the classes $[\theta_\alpha], [\Theta_{\check\alpha}] \in H^*(V)$ and $[d\phi_{\bar\gamma}] \in H^*(S^1)$, abstract polyfold perturbations and sequences of coherent collections of sections $(s_j^\pm)$, there exist an isomorphism

$$d\varphi^\pm : H(\mathcal{T}^{(k,l)}\mathbf{V}_0^+, \mathcal{L}_{X^{0+}}) \to H(\mathcal{T}^{(k,l)}\mathbf{V}_0^-, \mathcal{L}_{X^{0-}})$$

which is simply the lift to the tensor algebra of the isomorphism

$$\varphi^\pm : H_*(\mathfrak{P}_0^+; d_0^+) \to H_*(\mathfrak{P}_0^-; d_0^-),$$

constructed in [EGH] by studying curves in the cobordims $W = \overrightarrow{V^+ V^-}$ interpolating between the two different choices (see also the discussion on invariance for satellites there).

Moreover the descendant hamiltonians $h_{\check\alpha,n}$ induce covariant (with respect to $d\varphi^\pm$) Hamiltonian vector fields $X_{\check\alpha,n} \in H_*(\mathcal{T}^{(1,0)}\mathbf{V}_0; \mathcal{L}_{X^0})$, $\check\alpha = 1, \ldots, L$, $n = 0, 1, 2, \ldots$.

Theorem 3.1.

$$\mathcal{L}_{X^0}\omega = 0$$

Proof. We proceed exactly as in the contact homology case, only keeping in mind that, this time, nodal configurations can appear in codimension 1 when studying the moduli spaces, relevant for N, of curves with a doubly S^1-constrained line joining the two special 0 and ∞ punctures. Indeed such extra boundary, containing nodal curves where the node separates the 0 and ∞ puncture, corresponds to the term $\omega^{A\mu} \frac{\partial (X^0)^B}{\partial t^\mu} \frac{\partial}{\partial v^A} \otimes \frac{\partial}{\partial v^B}$ and $\frac{\partial (X^0)^A}{\partial t^\mu} \omega^{\mu B} \frac{\partial}{\partial v^A} \otimes \frac{\partial}{\partial v^B}$ in the Lie derivative $\mathcal{L}_{X^0}\omega$. $\square$

3.2. **Descendant Hamiltonian vector fields and ω-recursion.** The following result is the analogue of Theorem 2.6 (and proved in completely similar way) for the rational SFT case, and shows how the non-equivariant bivector ω is related to the geometry of gravitational descendants and the combined knowledge of differential of the Hamiltonian $d\mathbf{h}_{\check\alpha,n} \in H_*(\mathcal{T}^{(0,1)}\mathbf{V}_0, \mathcal{L}_{X^0})$ and of the graded symmetric bivector $\omega \in H_*(\mathcal{T}^{(2,0)}\mathbf{V}_0, \mathcal{L}_{X^0})$ allows to recover the descendant vector fields $X_{\alpha,n+1} \in H_*(\mathcal{T}^{(1,0)}\mathbf{V}_0, \mathcal{L}_{X^0})$, $n \geq 0$. Notice however how, in general, this is not equivalent to recovering the Hamiltonians $h_{\alpha,n+1}$ themselves.

Theorem 3.2.

$$X_{\check\alpha,n+1} = \Pi(\cdot, d\mathbf{h}_{\check\alpha,n+1}) = \omega(\cdot, d\mathbf{h}_{\check\alpha,n}) \qquad \in H_*(\mathcal{T}^{(1,0)}\mathbf{V}_0, \mathcal{L}_{X^0})$$

Proof. The statement is proved precisely in the same way as for Theorem 2.6. Notice only that the analogue of the term containing the constants $C_{\alpha,k}^\mu$, counting nodal curves, in this case is absorbed in the Lie derivative that vanishes in homology.

$\square$

Notice that the above recursion makes sense for $n = -1$ too if we define $\mathbf{h}_{\breve{\alpha},-1} := \eta_{\breve{\alpha}\beta}t^{\beta}$. Then all of our sequences of Hamiltonians $\mathbf{h}_{\breve{\alpha},n}$ satisfy to a recursion which starts from a Casimir at level $n = -1$. This allows to deduce commutativity $\{\mathbf{h}_{\breve{\alpha},i}, \mathbf{h}_{\breve{\beta},j}\} = 0$, which we know to hold on homology, simply from the recursion, since

$$\begin{aligned}
\{\mathbf{h}_{\breve{\alpha},i}, \mathbf{h}_{\breve{\beta},j}\} &= \omega(d\,\mathbf{h}_{\breve{\alpha},i}, d\,\mathbf{h}_{\breve{\beta},j-1}) = \\
&= -\{\mathbf{h}_{\breve{\alpha},i+1}, \mathbf{h}_{\breve{\beta},j-1}\} = \\
&= \ldots = \\
&= (-1)^{j+1}\{\mathbf{h}_{\breve{\alpha},i+j+1}, \mathbf{h}_{\breve{\beta},-1}\} = 0.
\end{aligned}$$

Example 3.3. Consider again the case $V = S^1$ with $t = t^1\theta_1 + \tau^1\Theta_1$, $\theta_1 = 1$ and $\Theta_1 = d\varphi$ where φ is the angular coordinate on S^1. Here, as in any other circle bundle over a symplectic manifold with even cohomology, $\mathbf{h}^0 = 0$ and everything happens at chian level. Even in the full rational SFT case, it is straightforward to compute ω. We write $\pm k$ for the index $(k\gamma, \pm)$ associated to the k-th multiple of the positive or negative orbit $\gamma = V$ and we use the index 0 to refer to the component along t^1 (or, in other words, $v^0 = t^1$). From the dimension formula for the moduli space of SFT-curves and some easy curve counting we immediately see that

$$\omega^{kl} = (k+l)v^{k+l}, \qquad k,l \in \mathbb{Z}$$

Applying ω-recursion we can recover the n-th descendant Hamiltonian. Indeed, let us start with

$$\mathbf{h}_{1,0} = \frac{1}{2}\sum_k v^{-k}v^k.$$

Recursion tells us

$$\begin{aligned}
\frac{\partial\,\mathbf{h}_{1,1}}{\partial v^j}\Pi^{jl} = l\frac{\partial\,\mathbf{h}_{1,1}}{\partial v^{-l}} &= \sum_k (k+l)v^{-k}v^{k+l} = \\
&= \frac{1}{2}\left(\sum_k (k+l)v^{-k}v^{k+l} + \sum_{k'}(-k')v^{k'+l}v^{-k'}\right) = \\
&= \frac{l}{2}\sum_k v^{-k}v^{k+l}
\end{aligned}$$

from which we deduce

$$\mathbf{h}_{1,1} = \frac{1}{6}\sum v^{-k}v^{k+l}v^{-l}.$$

Notice that, actually, for $l = 0$, the above equation is void, as we expected. The same procedure can be reiterated to find

$$\mathbf{h}_{1,n} = \frac{1}{n!}\sum_{k_1+\ldots+k_n=0} v^{k_1}\ldots v^{k_n}.$$

$\triangle$

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